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MODE-RAY DUALITY

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ABSTRACT

Earlier results in the theory of terrestrial radio waves are applied to seismology. A partial field of the complete eigen-value solution for a sphere can be interpreted as real rays. Watson's transformation and the *WKBJ* approximation are employed to establish links between the index trio (l, m, n) of a mode and the corresponding parameters of the ray trajectory associated with this mode. It is shown that Snell's law for rays and Jeans' formula are complementary. The condition of constructive interference is expressed as an integral equation for the eigen-frequencies ${}_n\omega_l$.

Introduction

Brune (1964) has recently derived a relation between the travel-times of certain seismic phases and the serial number of the associated normal mode. His results reflect in a very simple manner the existing duality between the two representations of the displacement field in a sphere, as normal modes and rays. Van der pol and Bremmer (1937) and Bremmer (1949) investigated this problem in great detail in connection with propagation of radio waves in the spherical wave-guide around the earth. In the present study we have amplified one of their results in order to obtain a correspondence between rays and normal modes in the earth's mantle.

Notation

A spherical coordinate system (R, θ, φ) is set up at the center of a spherical earth with radius $R = a$. The basic solutions of the scalar wave equation $\nabla^2\psi + K^2\psi = 0$ in this system are $\psi_{lmn} = z_l({}_nK_lR)W_l^m(\cos\theta)e^{\pm im\varphi}$. In our case $z_l({}_nK_lR)$ refers to the spherical Bessel functions of the first kind, of order l, and $W_l^m(\cos\theta)$ stands for the associated Legendré function of the first kind, order l, and degree m. $\mathrm{Re}\{W_l^m\} = p_l^m(\cos\theta)$ $\mathrm{Im}\{W_l^m\} = q_l^m(\cos\theta)$ (definition as in Jeffreys and Jeffreys, 1956). The symbol ${}_n\omega_l$ will designate the angular frequency of the n-th normal mode belonging to the order l $(n = 1, 2, 3 \cdots)$. K is the wave number as usual.

Separation of the Non-diffracted Body Waves

There are several ways of representing the total displacement field in a sphere. The normal mode expansion is straightforward since it arises naturally from the formulation of the boundary value problem in spherical coordinates. To obtain this representation one performs a Fourier transform over frequency for each component of the field and then evaluates the residues at the poles of the transfer function in the complex frequency plane (e.g. Gilbert and MacDonald 1960). The general form of the time response of a layered sphere to a source localized both in time and space can be written in the form:

$$U(t) = F_0 \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)}\, p_l^m(\cos\theta) \sum_{n=1}^{\infty} \frac{N({}_n\omega_l, l)}{\dfrac{\partial}{\partial\omega}\{F({}_n\omega_l, l\}_{\omega={}_n\omega_l}} \sin({}_n\omega_l\, t) \tag{1}$$

where F_0 is a source function and N/F is the medium response function. It was assumed in (1) without loss of generality that m is constant.

The explicit solution in equation (1) converges very slowly for $l < n < {}_nK_l a$. The original sums in the frequency domain are therefore transformed into integrals by means of Watson's transformation. Suppressing the summation over l we can write for each component of the displacement field,

$$U(\omega) = -i \int_{L_1} \frac{l\,dl}{\cos \pi l} f\left(l - \frac{1}{2}\right) p_{l-1/2}(-\cos\theta) \tag{2}$$

where the contour L_1 encloses the real semi-axis $l > 0$ and avoids the real roots of $f(l - \frac{1}{2})$. After the separation of the surface waves the contour L_1 encloses the entire real semiaxis since there are no more real roots of $f(l - \frac{1}{2})$ on this axis. It is easy to see that L_1 is equivalent to the contour $-\infty - i\epsilon$, $\infty - i\epsilon$ (ϵ—an arbitrarily small number) which is a straight line below the real axis, provided that $f(l - \frac{1}{2})$ is an even function of l. In order to render $f(l - \frac{1}{2})$ even in l and at the same time give a *clear physical interpretation to the Watson integral* (2), Van der pol and Bremmer (1937), Jeffreys and Lapwood (1957), Scholte (1956) converted it into an expansion of the form

$$U(\omega) = \sum_{j=0}^{\infty} U^{(j)} = i \sum_{j=0}^{\infty} \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{(l + \frac{1}{2})}{\sin \pi l} f^{(j)}(l) p_l\{\cos(\theta - \pi j)\}\, dl \tag{3}$$

known as the "rainbow expansion".

The coefficients $f^{(j)}$ are obtained in the following way: first one splits the nominator of $f(l)$ into four groups of terms, depending on their being ingoing, outgoing, longitudinal and shear. Then the direct waves are separated ($j = 0$) and the rest is expanded in geometrical series. The detailed procedure can be found in Van der pol and Bremmer (1937). We next make use of the identity

$$p_l(-\cos\theta) = e^{-i\pi l} p_l(\cos\theta) + i \sin \pi l \left\{ p_l(\cos\theta) + i q_l(\cos\theta) \right\} \tag{4}$$

and further split the body-wave field into two parts,

$$\begin{aligned} U^{(body)} = {} & -2\pi e^{i\omega t} \sum_{l_s} \frac{(l + \frac{1}{2})}{\sin \pi l_s} \frac{N(l_s, \omega)}{\left[\dfrac{\partial F(l, \omega)}{\partial l}\right]_{l=l_s}} e^{-i\pi l_s} p_{l_s}(\cos\theta) \\ & - e^{i\omega t} \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} (l + \tfrac{1}{2}) f^{(j)}(l, \omega) \left\{ p_l(\cos[\theta - j\pi]) + i q_l(\cos[\theta - j\pi]) \right. \end{aligned} \tag{5}$$

The first term has been expanded again into residue series.

This sum is similar in form to the surface wave series, but it is evaluated at l_s, the complex poles of the period equation in the lower half of the l plane. For distances $\Delta = a\theta$ which are not too small the convergence will be very rapid due to exponential attenuation factor $e^{-\theta|\mathrm{Im}\, l_s|}$. It seems therefore plausible that these terms correspond to diffracted body waves which propagate in the azimuthal direction. It can be shown that the attenuation coefficient of these waves is proportional to one-third power of the frequency. The rainbow integrals have no poles in the complex l-plane and can therefore be taken over the entire real l axis. This path passes through saddle points obtained by the introduction of the asymptotic approximations for the participating eigenfunctions. However, in order to be able to express our final results in terms of normal modes we shall temporarily re-sum over j and then perform an integration in the ω-plane. Omitting henceforth th· ·ffracted field we obtain the body-wave field in the form

$$U(t) = -i \sum_{n=0}^{\infty} e^{i_n\omega_l t} \int_{-\infty}^{\infty} (l + \tfrac{1}{2}) f({}_n\omega_l, l) \left\{ p_l(\cos\theta) + i q_l(\cos\theta) \right\} dl \tag{6}$$

Snell's Law

We wish next to perform the integration over the real l axis under the condition $l < n < {}_nK_l a$. To this end we shall need the asymptotic approximation for $h_l^{(2)}({}_nK_l R)$ and $p_l(\cos\theta)$. These are known to be (Morse and Feshbach 1953).

$$h_l^{(2)}({}_nK_l R) = \frac{\exp -i\left[\sqrt{{}_nK_l^2 R^2 - (l+\frac{1}{2})^2} - (l+\frac{1}{2})\cos^{-1}\left\{\frac{l+\frac{1}{2}}{{}_nK_l R}\right\} - \frac{\pi}{4}\right]}{{}_nK_l R\sqrt{\sin\alpha}} + 0\{({}_nK_l R \sin\alpha)^{-3/}\tag{7}$$

$$l > 1,\ {}_nK_l R \gg 1,\ \frac{l+\frac{1}{2}}{{}_nK_l R} = \cos\alpha = \sin\tau_n,\ 1 - \cos\alpha > \frac{3}{{}_nK_l R}\sqrt[3]{l+\tfrac{1}{2}}$$

$$p_l^m(\cos\theta) + i q_l^m(\cos\theta) = \frac{\exp -i\left[(l+\frac{1}{2})\theta - \frac{m\pi}{2} - \frac{\pi}{4}\right]}{\sqrt{\pi\sin\theta}}\sqrt{2}l^{m-1/2} + 0(l^{m-3/2}) \tag{8}$$

$$\epsilon \leqq \theta \leqq \pi - \epsilon,\ \epsilon > 0,\ l \gg m,\ l \gg \frac{1}{\epsilon}$$

Thus we obtain for the asymptotic form of an eigenfunction

$$-ie^{i_n\omega_l t} h_l^{(2)}({}_nK_l R)\left[p_l^m(\cos\theta) + i\, q_l^m(\cos\theta)\right] \simeq \frac{\sqrt{2}\, l^{m-1/2}}{{}_nK_l R\sqrt{\pi\sin\theta\cos\tau_n}} \cdot \exp i\left[{}_n\omega_l t - {}_nK_l R\{\cos\tau_n - (\alpha-\theta)\sin\tau_n\} + m\frac{\pi}{2}\right] \tag{9}$$

The constant phase $\phi_0 = m(\pi/2)$ arises from the multipolarity of the source.

Following Bremmer (1949) we interpret the phase as follows: consider a sphere centered at 0 (Fig. 1) with radius ${}_nr_l = (l + \frac{1}{2})/{}_nK_l = R \sin \tau_n$. It is then clear from the geometry of this figure that $SP \simeq PQ - QS'$, $QS' = \text{arc}(\alpha - \theta)$. Therefore the quantity $R[\cos \tau_n - (\alpha - \theta) \sin \tau_n] = D$ is approximately equal to the ray path from the source S to the station P provided that the angle τ_n is interpreted as the angle of incidence of the ray at the surface. The ray constant p then becomes,

$$p({}_n\omega_l) = a \frac{dt_p({}_n\omega_l)}{d\Delta} = \frac{R \sin \tau_n}{v} = \frac{l + \frac{1}{2}}{{}_n\omega_l} = \frac{a}{c} \tag{10}$$

where v is the velocity, a is the radius of the sphere, $t_p({}_n\omega_l)$ is the phase travel-time along the ray and ${}_n\omega_l$ is the angular frequency of the normal mode. The important feature of equation (10) is that it ties up the ray scheme (Snell's law) with mode scheme (Jeans' formula) and thus associates the order of the free oscillation with the angle of incidence. Furthermore, this relation is invariant to radial heterogenity provided that the first term in the saddle point approximation is sufficient, that is to say, whenever the 'geometrical-optics' approximation is valid.

The interpretation of equation (10) is simple: The sphere ${}_nr_l = (l + \frac{1}{2})/{}_nK_l$ is considered as an envelope of rays. If the point of observation P is inside this sphere, the path length D will be complex such that $\text{Im}\, D < 0$ and the wave motion will be dampened exponentially. This means that the energy associated with the pair $(l, {}_n\omega_l)$ will travel along a localized path which we call a ray. Each given pair defines its own radius and its own path. The same ray may, however, be shared by different frequencies which belong to the same radius, ${}_nr_l$. If, on the other hand, we fix the point of observation on the surface of the sphere and choose the pairs $(l, {}_n\omega_l)$ arbitrarily we shall face three possibilities:

1. $(l + \frac{1}{2}) < {}_nK_la$ or ${}_nr_l < a$. The angle of incidence is real and the ray may exist.
2. $(l + \frac{1}{2}) > {}_nK_la$ or ${}_nr_l > a$. No real rays exist.
3. $(l + \frac{1}{2}) = {}_nK_la$ or ${}_nr_l = a(\tau_n = \pi/2)$. The sphere's surface coincides with the caustic (ray envelope). Additional term of the saddle-point approximation is needed to obtain a finite field at the caustic.

The previous results remain valid for a radially heterogeneous sphere, provided, the heterogenity is small over a wave-length. Using the well known *WKBJ* method, we may write the solutions of the radial differential equation as

$$f_n(R) = \left(\frac{v_0^2}{v^2(R)} - \frac{l(l+1)}{K_0^2R^2}\right)^{-1/4} \exp \pm i_n\omega_l \int \sqrt{\frac{1}{v^2(R)} - \frac{l(l+1)}{{}_n\omega_l^2R^2}}\, dR(s) \tag{11}$$

where $v_0 = v(a)$, $K_0 = {}_n\omega_l/v_0$ ${}_nK_l = \frac{{}_n\omega_l}{v(R)}$ and s is the ray-length parameter.

It then follows that

$$f_n(R) = \frac{\exp -i\left[\int_{{}_nr_l}^{R} \sqrt{{}_nK_l^2R^2 - l(l+1)}\, \frac{dR}{R} - \frac{\pi}{4}\right]}{\sqrt{K_0R[l(l+1) - {}_nK_l^2R^2]^{1/2}}} \qquad R > {}_nr_l \tag{12}$$

$$f_n(R) = \frac{\exp -\int_{R}^{{}_nr_l} \sqrt{l(l+1) - {}_nK_l^2R^2}\, \frac{dR}{R}}{\sqrt{K_0R[l(l+1) - {}_nK_l^2R^2]^{1/2}}} \qquad R < {}_nr_l \tag{13}$$

If l is given and the point of observation is allowed to move freely, we will find an exponential decay below the critical radius and an oscillatory motion above it. This is the physical picture of the ray.

The Phase Integral and the Phase Travel-time

The mode-ray correspondence requires also that the radial mode number n will take part in the ray regime. Consider for example, a toroidal mode ${}_nT_l$ in the earth's mantle. This mode, which has $n - 1$ zero crossings in the radial direction exists due to a standing wave pattern in a spherical wave guide. This pattern is sustained by a constructive interference of ingoing waves reflected at the surface with outgoing

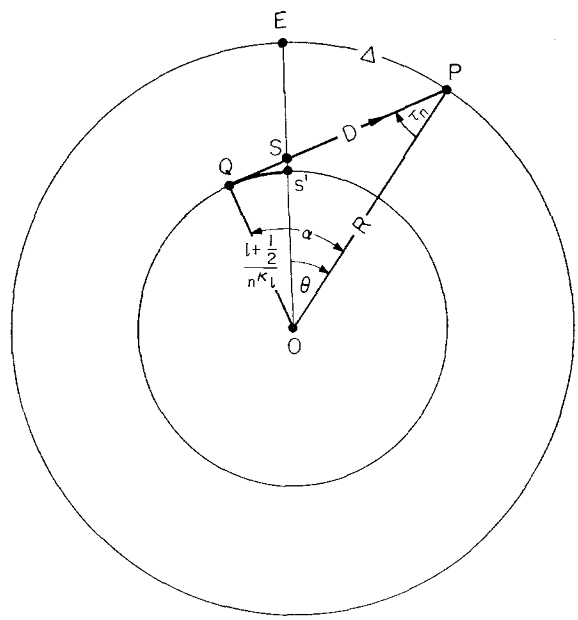

Fig. 1. Geometrical interpretation of the saddle-point approximation for the spherical wave

$$h_l^{(2)}({}_nK_lR)[p_l(\cos\theta) + iq_l(\cos\theta)].$$

waves reflected at the level $R = {}_nr_l$ (Fig. 1). We have seen that for values of l and n such that $l < n < {}_nK_lR$ a ray picture is adequate. The cumulative radial phase as measured by an observer moving along the ray from the caustic level $R = {}_nr_l$ up to the surface is given by the integral $\int_{{}_nr_l}^{a} \sqrt{{}_nK_l{}^2R^2 - l(l+1)}\, dR/R$. In the mode regime this change of phase is $(n - 1)\pi$. [The phase $\pi/4$ disappears eventually: see equation (9)]. Hence the so-called *phase-integral relation* (Bremmer 1949).

$$\int_{{}_nr_l}^{a} \sqrt{\eta^2 - \frac{l(l+1)}{{}_n\omega_l{}^2}}\,\frac{dR}{R} = \frac{\pi(n-1)}{{}_n\omega_l} + \phi_r \tag{14}$$

where $\eta = R/v(R)$ and ϕ_r is the phase change, upon reflection at the free surface.

Note that for a given velocity-depth profile $v(R)$ and for a given pair (l, n), equation (14) is an integral-equation for ${}_n\omega_l$.

The travel-time t_p of a direct ray is given by ray-theory (Bullen 1954)

$$t_p = p\theta + 2\int_{r_0}^{a} \sqrt{\eta^2 - p^2}\,\frac{dR}{R} \tag{15}$$

p is the ray constant and r_0 is the radial distance to the lowest point of the ray. Invoking the ray-mode correspondence relations as given in equations (10) and (14) we obtain for the phase travel-time

$$t_p = \Delta\frac{dt_p}{d\Delta} + \left(n - 1 + \frac{m}{4} + \phi_r + \phi_0\right){}_nT_l/2\pi \tag{16}$$

where ϕ_0 is the initial phase of the source. For $m = 0$ this equation becomes identical to that of Brune (1964).

Equation (16) can be considered as a differential equation of the Clairaut type for t_n. Its general solution is

$$t_p = \Delta/c + \left(n - 1 + \frac{m}{4} + \phi_r + \phi_0\right){}_nT_l/2\pi \tag{17}$$

in accord with equation (10).

It is clear from equation (10) that each mode will travel along its own ray with its own constant phase velocity. Other modes may travel along the same ray *only* if they have a common ratio $(l + \frac{1}{2})/{}_n\omega_l$. Unless this is the case they will move along *different* rays and arrive to a surface station at times given by equation (16).

Assume next that the neighborhood of ${}_n(\omega_0)_l$ is sufficiently dense so that one may think of the angular frequency as a continuous variable. As a result, the motions in neighboring rays will interfere and *wave-packets* will be formed. One may thus define group arrival, group delay and group velocity analogous to the case of the surface waves. All the necessary expressions can be derived directly from equations (10) and (16) by differentiation with respect to ω. Hence for example, the definition of the group velocity U by means of the group delay t_g

$$t_g = \frac{\partial\phi(\omega)}{\partial\omega} = \frac{\Delta}{U} - t_p \tag{18}$$

where the derivative of the phase is taken at constant mode number n, analogous to the definition of the group velocity of surface waves.

Conclusion

The application of the foregoing equations is limited to frequency bands for which the condition $l < n < {}_nK_l a$ holds. Outside this range the normal mode regime is dominant. Special approximations can be applied to the transition region $l \simeq {}_nK_l a$ In the range $l < n < {}_nK_l a$ equation (10) will always be meaningful. Although the validity of equation (14) was not demonstrated for spheroidal modes equation (16) will still hold since the condition for constructive interference has the same form.

The distribution of the various modes in the different rays (such as *pP*, *P*, *S*, *SS*, *SKS*, *PKP*, etc.) is left as an open question. The "rainbow expansion" mentioned earlier is the key to the solution of this important problem.

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